

RANDOM STRATEGIES WITH HISTORICAL MEMORY FOR THE ROBIN HOOD GAME

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ABSTRACT. The *Robin Hood* game is played as follows: On day i , the Sheriff puts $s(i)$ bags of gold in the cave. On night i , Robin removes $r(i)$ bags from the cave. The game is played for each $i \in \mathbb{N}$. Robin wins if each bag which was put in the cave is eventually removed from it; otherwise the Sheriff wins.

Gasarch, Golub, and Srinivasan studied the Robin Hood game in the case of random strategies where Robin has no historical memory. We extend their main result to the case of bounded historical memory, and obtain a hierarchy of provably distinct games.

1. THE ROBIN HOOD GAME

The *Robin Hood* game $RH(r, s, A)$ is defined for functions $r, s : \mathbb{N} \rightarrow \mathbb{N}$ such that $1 \leq r(i) < s(i)$ for each i , and for a set A , as follows:

- (1) On day i , the *Sheriff* (of Nottingham) puts $s(i)$ bags of gold in the cave, each labelled by an element of A . No label is used twice (over the course of the entire game).
- (2) On night i , *Robin* (Hood) removes $r(i)$ bags from the cave.

The game is played for each $i \in \mathbb{N}$. Robin wins if each bag which was put in the cave is eventually removed from it; otherwise the Sheriff wins.

It is easy to see that if Robin has an unlimited historical memory (knowing at each night i which of the bags in the cave appeared first), then he has a winning strategy: On night i pick $r(i)$ bags out of those which arrived first.

Deterministic strategies for this game were studied, from the set-theoretic point of view, in [2, 3]. Gasarch, Golub and Srinivasan [1] consider the case where Robin has *no historical memory*, that is, he cannot distinguish between the days where the bags were put in the cave. They suggest the following *probabilistic* strategy for Robin: On night i , remove random $r(i)$ bags out of the cave (with uniform distribution). They say that Robin wins *almost surely* if for each bag put in

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the cave, its probability of being eventually removed is 1. The probability is taken over Robin's coin tosses. More precisely, the probability that a bag is not removed is the product of all probabilities p_i that the bag is not removed at night i , and the Sheriff's winning (or Robin's loosing) probability is the supremum of all probabilities p_x that a bag x put in the cave is never removed.¹ Let $L(i) = \sum_{j=1}^i s(j) - r(j)$ denote the number of bags in the cave after night i . The main result in [1] is that Robin wins almost surely if, and only if, the series

$$\sum_{i=1}^{\infty} \frac{r(i)}{L(i) + r(i)}$$

diverges; otherwise Robin loses almost surely.

2. STRATEGIES WITH BOUNDED HISTORICAL MEMORY

We generalize the above mentioned result to the case of bounded historical memory. The typical case is that Robin can, on each night, identify the bags put in the cave on the last k days, where k is constant. It will turn out that already the natural strategy for historical memory $k = 1$ is strictly stronger than the natural strategy in the memoryless case. Moreover $k = 2$ yields a strictly stronger strategy than $k = 1$, etc. (Theorem 3.1). In fact, the game can be analyzed in a much broader family of cases, as will be shown in the sequel.

The most general case is that Robin can, on night i , identify the bags put in the cave on the last $b(i)$ days, where $b : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$ is a function with $b(i) \leq i$ for all i .² (So that $b(i) \equiv 0$ is the memoryless case studied in [1]). It is natural to denote this game by $RH(r, s, b, A)$, but our analysis below is independent of the set A , so we will simply write $RH(r, s, b)$. A key observation is that the following natural restriction leads to a substantial simplification of the analysis of the generalized games.

Restriction 2.1. We pose the restriction that Robin cannot remember anything that he forgot earlier, that is, $b(i + 1) \leq b(i) + 1$ for each i ; equivalently, the function $i - b(i)$ is nondecreasing.

We suggest the following deterministic and random strategies for Robin, motivated by the strategy given in [1] for the memoryless case: Call a bag *very old* if Robin cannot tell the day it was put in the cave.

¹Our definition of the Sheriff's winning probability is simpler than the one given in [1], but both our and the proofs of [1] work for both definitions and actually imply that the definitions are equivalent.

² $b(i)$ stands for the *bound* on Robin's historical memory on night i .

An important observation is that Robin can identify the very old bags since he can identify the bags which are *not* very old.

Oldest_{DET}: On night i Robin chooses any $r(i)$ many bags out of the very old bags. If there are less than $r(i)$ many very old bags, then Robin also chooses some of the oldest bags among the ones he remembers, so as to choose $r(i)$ bags in total.³

Oldest_{RND}: Same as **Oldest_{DET}**, but the $r(i)$ bags are chosen at random, with uniform probability, out of the older bags.⁴

Observe that if **Oldest_{DET}** is a winning strategy for Robin, then so is **Oldest_{RND}**.

Write

$$\tilde{L}(i) = \max \left\{ 0, \sum_{j=1}^{i-b(i)} s(j) - \sum_{j=1}^{i-1} r(j) \right\}$$

Then $\tilde{L}(i)$ is the number of bags put in the cave on days $1, 2, \dots, i-b(i)$ and not removed until day i .

Proposition 2.2.

- (1) *If $i - b(i)$ is bounded, then **Oldest_{DET}** is a winning strategy for Robin in $RH(r, s, b)$ (for each r , and s).*
- (2) *If there exist infinitely many i such that $\tilde{L}(i) \leq r(i)$, then **Oldest_{DET}** is a winning strategy in $RH(r, s, b)$.*

Proof. (1) is easy. To prove (2), assume that a bag was put in the cave on day d . By (1) we may assume that $i - b(i)$ is unbounded. Let i be such that $d < i - b(i)$. Restriction 2.1 ensures that this will also hold for all larger i 's, so we may assume further that $\tilde{L}(i) \leq r(i)$. This means that on *night* i , all bags put on days $\leq i - b(i)$ (in particular, those put on day d) were removed from the cave. \square

We may therefore make the following additional restriction.

Restriction 2.3. $\tilde{L}(i) > r(i)$ for all but finitely many i .

³To put this more precisely, on night i Robin has a partition of the bags in the cave into disjoint (possibly empty) sets $S_0, S_1, \dots, S_{r(i)}$ such that S_0 is the set of very old bags, and for $k = 1, \dots, r(i)$, S_k is the set of bags put in the cave $r(i) - k$ days ago. If $|S_0| \geq r(i)$, then Robin chooses any $r(i)$ many bags out of the bags in S_0 . Otherwise, let m be the minimal such that $r(i) < \sum_{k=0}^m |S_k|$. Then Robin takes all bags in the sets S_0, \dots, S_{m-1} , as well as $r(i) - \sum_{k=0}^{m-1} |S_k|$ many bags from S_m .

⁴Using the notation of the previous footnote, If $|S_0| \geq r(i)$, then Robin chooses at random (with uniform probability) $r(i)$ many bags out of the bags in S_0 . Otherwise Robin takes all bags in the sets S_0, \dots, S_{m-1} , as well as $r(i) - \sum_{k=0}^{m-1} |S_k|$ many random bags from S_m .

Theorem 2.4. *Assume that r , s , and b satisfy Restrictions 2.1 and 2.3, and Robin uses the strategy $\text{Oldest}_{\text{RND}}$.*

- (1) *If $\sum r(i)/\tilde{L}(i) = \infty$, then Robin wins almost surely.*
- (2) *If $\sum r(i)/\tilde{L}(i) < \infty$, then the Sheriff wins almost surely.*

Proof. If there is i such that $\tilde{L}(i) \leq r(i)$, let i^* be the maximal such i . Then on night i^* , all bags put in the first $i^* - b(i^*)$ days are removed, and $\tilde{L}(i) > r(i)$ for all $i > i^*$. Since the convergence of the series in question does not depend on the first few elements, this shows that we may assume that $\tilde{L}(i) > r(i)$ for all i .

Observe that if $i - b(i)$ is bounded, then $\tilde{L}(i)$ is eventually equal to 0, contradicting our assumption, thus $i - b(i)$ is unbounded. Assume that a bag was put in the cave on day d , then for each large enough i , $d < i - b(i)$ so that on night i , the probability that the bag in question is removed is $r(i)/\tilde{L}(i)$.

Now, the probability that a bag put in the cave on day d is *not* eventually removed is

$$(1) \quad \prod_{i=d}^{\infty} \left(1 - \frac{r(i)}{\tilde{L}(i)}\right).$$

The product (1) converges to 0 (i.e., Robin wins almost surely) if $\sum r(i)/\tilde{L}(i) = \infty$. If $\sum r(i)/\tilde{L}(i) < \infty$, then the product (1) is positive for $d = 1$. Thus, its limit when $d \rightarrow \infty$ is 1. Consequently, the Sheriff wins almost surely. \square

Note that in Theorem 2.4, (1) implies that the other direction of (2) also holds, and (2) implies that the other direction of (1) also holds. Thus, the theorem gives an exact characterization of when Robin wins almost surely and when the Sheriff wins almost surely, and shows that it is always the case that one of them wins almost surely. In the case $b(i) \equiv 0$, Theorem 2.4 reduces to the main result of [1], described at the end of Section 1.

3. HISTORICAL MEMORY HELPS

To make sure that the generalization made in 2.4 is not trivial, we must find instances where additional historical memory changes the strategy's status from a losing strategy to a winning one.

Assume that $b, c : \mathbb{N} \rightarrow \mathbb{N}$. We say that c *eventually dominates* b if there exists m such that for all $i > m$, $b(i) < c(i)$.

Theorem 3.1. *Assume that c eventually dominates b , $i - c(i)$ is unbounded, and b and c satisfy Restrictions 2.1 and 2.3. Then there exist*

functions $r, s : \mathbb{N} \rightarrow \mathbb{N}$ such that for Robin, Oldest_{RND} is a (surely) winning strategy in $RH(r, s, c)$ and an almost-surely losing strategy in $RH(r, s, b)$.

Proof. Since convergence of series does not depend on the first few terms, we may assume that for each i , $b(i) < c(i)$. Since $\tilde{L}(i)$ also depends on the bounding function b , let us denote it here by $\tilde{L}_b(i)$. It follows that for each r and s , $\tilde{L}_c(i) < \tilde{L}_b(i)$ for all i . It thus suffices to consider the case where $c(i) = b(i) + 1$ for each i , and therefore $\tilde{L}_c(i) + s(i - b(i)) = \tilde{L}_b(i)$.

At step i of the construction we have the definition of s at $1, \dots, i - c(i)$ and r at $1, \dots, i - 1$, and therefore $\tilde{L}_c(i)$ is defined. Define $r(i) = \max\{i, \tilde{L}_c(i)\}$, and define s on $i - c(i) + 1, \dots, i + 1 - c(i + 1)$ to be $r(i)^3$ (so that $r(i)/\tilde{L}_b(i) = r(i)/(\tilde{L}_c(i) + s(i - b(i))) = r(i)/(r(i) + r(i)^3) < 1/r(i)^2 \leq 1/i^2$).

Since $r(i) \geq \tilde{L}_c(i)$ for each i , we have by Proposition 2.2 that Oldest_{RND} is a winning strategy in $RH(r, s, c)$. Now, $\sum r(i)/\tilde{L}_b(i) \leq \sum 1/i^2 < \infty$, so by Theorem 2.4, Oldest_{RND} is an almost-surely losing strategy in $RH(r, s, b)$. \square

In particular, we have the following.

Corollary 3.2. *For each $n = 0, 1, 2, \dots$, there exist functions $r, s : \mathbb{N} \rightarrow \mathbb{N}$ such that for Robin, Oldest_{RND} is a (surely) winning strategy in $RH(r, s, n + 1)$ and an almost-surely losing strategy in $RH(r, s, n)$.*

4. RANDOM SHERIFF

The authors of [1] pose the question of the behavior of the Robin Hood game when the Sheriff's strategy is random as well. Our analysis in Section 2, being independent of the Sheriff's moves, shows that the results apply to this case as well. In addition to the first strategy of [1] which was described in the introduction to the present paper (Section 1), a second random strategy for Robin is sketched in [1], and is conjectured to be an almost-surely winning strategy in the game $RH(1, s, [0, 1])$ where s is constant.

While we are unable to analyze the second strategy for lack of some details, we can see that the first strategy already works. Here $b(i) = 0$ (no historical memory) and $L(i) = \sum_{j=1}^i s(j) - r(j) = \sum_{j=1}^i s - 1 = i(s - 1)$, therefore

$$\sum_{i=1}^{\infty} \frac{r(i)}{\tilde{L}(i)} = \sum_{i=1}^{\infty} \frac{r(i)}{L(i) + r(i)} = \sum_{i=1}^{\infty} \frac{1}{L(i) + 1} = \sum_{i=1}^{\infty} \frac{1}{i(s - 1)} = \infty,$$

thus by Theorem 2.4, Oldest_{RND} is an almost-surely winning strategy in this game. Since Oldest_{RND} coincides with the first strategy of [1], we have that the first strategy of [1] is also an almost-surely winning strategy against a *random* Sheriff in $RH(1, s, [0, 1])$ (as well as any game $RH(r, s, A)$ with $\sum r(i)/(L(i) + r(i))$ diverging). Theorem 2.4 is an extension of this phenomenon to the case of nonzero historical memory.

5. OPEN PROBLEMS

Among the problems which naturally arise, the following two seem to be the most interesting.

Conjecture 5.1. *Oldest_{RND} is the best random strategy among the strategies which are independent of the index set A .*

Our analysis repeatedly uses Restriction 2.1 posed on the bounding function b .

Problem 5.2. *Analyze the case where b does not satisfy Restriction 2.1.*

Finally, our strategy Oldest_{RND} uses unboundedly many “random bits”. The referee suggests the problem whether good strategies exist, in which the number of random bits used at each specific step is bounded by some constant.

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